Maps Between Surfaces

Outline

1. Linear Transformations

If V and W are vector spaces, a **linear transformation** from V to W is a function $T\colon V \to W$ such that

1. $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$, and
2. $T(\lambda\vec{v}) = \lambda\, T(\vec{v})$

for all $\vec{v}_1, \vec{v}_2, \vec{v} \in V$ and $\lambda \in \mathbb{R}$. Linear transformations $\mathbb{R}^m \to \mathbb{R}^n$ correspond to $n \times m$ matrices, but linear transformations between other vector spaces don't correspond to matrices in any natural way.

2. Differentials of Maps

A **map between surfaces** is a differentiable function $f\colon S_1 \to S_2$, where S_1 and S_2 are regular surfaces. (We will write f instead of $\vec{f}$ to help the notation look a little cleaner.) The **differential** of such a map at a point $p \in S_1$ is a linear transformation

$$df_p\colon T_pS_1 \to T_{f(p)}S_2.$$

That is, df_p takes as input a tangent vector $\vec{t}$ at p, and outputs a corresponding tangent vector $df_p\big(\vec{t}\,\big)$ at $f(p)$.

The differential df_p can be defined using curves. Given a tangent vector $\vec{t}$ at p, let $\vec{x}(t)$ be a curve on S_1 such that $\vec{x}(a) = p$ and $\vec{x}\,'(a) = \vec{t}$. Then

$$df_p\big(\vec{t}\,\big) \;=\; \vec{y}\,'(a),$$

where $\vec{y}(t) = f\big(\vec{x}(t)\big)$ is the corresponding curve on S_2.

Note: We won't actually use the definition of a differential very often. In most cases, it is much easier to compute with differentials using a parametrization, as described below.

3. Differentials using Parametrizations

Let $f\colon S_1 \to S_2$ be a map between surfaces, and let $\vec{X}\colon U \to S_1$ be a parametrization. Let $\vec{F}\colon U \to S_2$ be the composition

$$\vec{F}(u,v) \;=\; f\big(\vec{X}(u,v)\big).$$

In this case, it follows from the chain rule that

$$df_p\big(\vec{X}_u\big) \;=\; \vec{F}_u \qquad \text{and} \qquad df_p\big(\vec{X}_v\big) \;=\; \vec{F}_v$$

for any point p. More precisely,

$$df_{\vec{X}(u,v)}\big(\vec{X}_u(u,v)\big) \;=\; \vec{F}_u(u,v) \qquad \text{and} \qquad df_{\vec{X}(u,v)}\big(\vec{X}_v(u,v)\big) \;=\; \vec{F}_v(u,v).$$

4. Types of Maps

Let $f\colon S_1 \to S_2$ be a map between two surfaces, let $\vec{X}\colon U \to S_1$ be a parametrization of S_1, and let $\vec{F}(u,v) = f\big(\vec{X}(u,v)\big)$.

1. We say that f is **equiareal** if $\|\vec{F}_u \times \vec{F}_v\| = \|\vec{X}_u \times \vec{X}_v\|$.
2. We say that f is an **isometry** if it is bijective and

$$\vec{F}_u \cdot \vec{F}_u = \vec{X}_u \cdot \vec{X}_u, \qquad \vec{F}_u \cdot \vec{F}_v = \vec{X}_u \cdot \vec{X}_v, \qquad \text{and} \qquad \vec{F}_v \cdot \vec{F}_v = \vec{X}_v \cdot \vec{X}_v.$$

3. We say that $\vec{f}$ is **conformal** if there exists a positive scalar $\lambda = \lambda(u,v)$ so that

$$\vec{F}_u \cdot \vec{F}_u = \lambda\, \vec{X}_u \cdot \vec{X}_u, \qquad \vec{F}_u \cdot \vec{F}_v = \lambda\, \vec{X}_u \cdot \vec{X}_v, \qquad \text{and} \qquad \vec{F}_v \cdot \vec{F}_v = \lambda\, \vec{X}_v \cdot \vec{X}_v.$$

Note that a bijective map is an isometry if and only if it is both equiareal and conformal.

5. The Jacobian

Let $f\colon S_1 \to S_2$ be a map between two surfaces, let $\vec{X}\colon U \to S_1$ be a parametrization of S_1, and let $\vec{F}(u,v) = f\big(\vec{X}(u,v)\big)$. Then the **Jacobian** of f at a point p is defined by the formula

$$Jf(p) \;=\; \frac{\|\vec{F}_u \times \vec{F}_v\|}{\|\vec{X}_u \times \vec{X}_v\|}.$$

More precisely,

$$Jf\big(\vec{X}(u,v)\big) \;=\; \frac{\|\vec{F}_u(u,v) \times \vec{F}_v(u,v)\|}{\|\vec{X}_u(u,v) \times \vec{X}_v(u,v)\|}.$$

Note that f is equiareal if and only if $Jf = 1$.

The Jacobian measures the area expansion of the map f around the point p. That is, if R_1 is an infinitesimal region around p with area dA_1, and $R_2 = f(R_1)$ is the corresponding region in S_2, then the area dA_2 of R_2 is given by the formula

$$dA_2 \;=\; Jf(p)\, dA_1.$$

More generally, if R_1 is any region in S_1 on which f is one-to-one, and $R_2 = f(R_1)$ is the corresponding region in S_2, then

$$\text{area}(R_2) \;=\; \iint_{R_1} Jf\, dA.$$